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A SUBSTITUTE FOR DUHAMEL'S THEOREM.

By Gilbert Ames Bliss.

Every teacher of the calculus, at some period in his course, finds himself facing the difficulties of deducing in an elementary way the expressions for geometrical or physical or mechanical quantities as definite integrals. The methods which he may use must be simple enough, in their rougher and more intuitive forms, to be comprehensible to beginners in the study of the subject, and yet they should be capable of refinement and precision without serious alteration, by filling in interstices, in order to satisfy the rigorous temperament of the more mature mathematician.

Two papers, by W. F. Osgood and R. L. Moore, explaining the use of Duhamel's theorem in this connection, have appeared in the *Annals*,* and have suggested to me the presentation of the present paper. I must confess that the theorem of Duhamel seems to me to be a somewhat clumsy and unsatisfactory instrument, though this is a subject upon which I find wide variations of opinion among my colleagues in various sections of the country. In its original form, and as it is at present stated in many of the elementary text books, the theorem conceals questions of uniform continuity, which are as essential in some form or other to the study of definite integrals as the notions of limits are to the differential calculus, and gives no intimation that something is lacking in the argument. This objection has been removed in the two papers referred to, but in a way which I find difficult to make intelligible to elementary students. For this reason I venture to lay before the readers of the *Annals* the theorem and methods of application explained in the following paragraphs. Needless to say I do not attempt to explain in an elementary course the notion of uniform continuity, notwithstanding its very great importance. But I try to leave a framework of proof in the mind of the student, of such a character that the moment he discovers uniform continuity, that moment he may also understand its application in the deduction of the theorem.

The theorem of § 1, which is the kernel of the paper, is in its essentials similar to those of Osgood and Moore, though different in form.† I do not hope to convince my readers at once of its utility as compared with that

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[*] Osgood, The integral as the limit of a sum and a theorem of Duhamel's, vol. 4 (1903), page 161; Moore, On Duhamel's theorem, vol. 13 (1912), page 161.

[†] The theorem and some of its applications were incorporated at my suggestion in a master's thesis by Mr. L. B. White entitled "Some applications of uniform continuity in the theory of definite integrals," The University of Chicago, 1910.

of Duhamel's theorem, though in my own courses it has seemed to be effective. But at least I should like to submit it for their consideration.

1. A Theorem Concerning Definite Integral Sums. Let us consider a function of the form $f(p, p', p'')$, where p is a symbol for a set of values (x, y, z), and p' and p'' for analogous sets. The points p, p', p'' are to range over a closed measurable region V in xyz-space in which f is continuous, and hence also uniformly continuous.

If the region V is divided into measurable sub-regions with maximum diameters less than δ and with volumes denoted by ΔV_k $(k = 1, 2, \cdots, n)$, and if in each region three points p_k, p_k', p_k'' are chosen, then

$$(1)\quad \operatorname*{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k', p_k'')\Delta V_k = \operatorname*{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k, p_k)\Delta V_k = \int_V f(p, p, p,)dV.$$

The proof of the theorem is very simple. Since f is uniformly continuous in V there corresponds to any positive constant ϵ a second positive constant δ such that

$$| f(p_k, p_k', p_k'') - f(p_k, p_k, p_k) | < \epsilon$$

whenever p_k, p_k', p_k'' lie in a sub-region ΔV_k with maximum diameter less than δ. Hence when V is divided into sub-regions with maximum diameters less than δ, as indicated above, the two sums whose limits are to be proved equal differ by a quantity less in absolute value than

$$\epsilon \sum_{k=1}^{n} \Delta V_k = \epsilon V,$$

provided that we let V denote the measure of the original region. The second limit in the formula (1) is by definition the definite integral indicated on the right, and so the theorem is proved.

The statement and proof of the theorem would apply exactly as they stand in case V were a region in one or two or any other number of dimensions; and it is clear that the method applies also if the number of points p selected in ΔV_k were more or less than three.

2. Applications of the Theorem. In the applications of the theorem of the preceding section two principles from the theory of functions of real variables are of great service; first, in any closed region a continuous function has a maximum and a minimum which it attains at points of the region; and second, if the region is connected, every value between the maximum and the minimum is assumed by the function at one point at least. These are statements which I find that younger students accept readily as being intuitively true; in fact, they usually regard one as being somewhat over particular if he dwells at length upon the necessity of proof. It is certain, however, that no one can penetrate far into a thorough understanding of the calculus without having convinced himself in some way or other of these properties of continuous functions.

The difficulties of the situation are not only mathematical ones, as Osgood has very clearly pointed out in his paper. It is necessary in many of the applications to presuppose properties of physical magnitudes which can not always conveniently be set down in advance. I shall therefore base a first application of the theorem as far as possible upon the properties of attractive force as stated by Osgood in his paper (page 175). The problem is to express as a definite integral the component parallel to the x-axis of the attraction of an arbitrary solid upon a particle of mass m situated exterior to the solid. It was considered by both Osgood and Moore in the papers referred to above.

The following notations are those of Osgood:

m = the mass of the fixed particle;
V = the attracting solid;
ρ = the density of V at a point $p(x, y, z)$;
α = the angle between the line mp and the x-axis.

The density ρ is supposed to be a continuous function of p.

Let V be divided into solids ΔV_k as in § 1 each of which fills out a closed measurable connected portion of space. For the portion ΔV_k the mass will be less than ΔV_k times the maximum of ρ in ΔV_k, greater than ΔV_k times the minimum of ρ, and hence equal to $\rho_k \Delta V_k$, where ρ_k is an intermediate value between the maximum and minimum and is surely taken on by ρ at some point p_k in ΔV_k. The attraction of ΔV_k for the particle m will similarly be expressible as $\kappa m \rho_k \Delta V_k / r_k'^2$, where κ is a gravitational constant and r_k' is an intermediate value between the maximum and the minimum of r in ΔV_k. The projection of the attraction on the x-axis is expressible in the form $\kappa m \rho_k \cos \alpha_k'' \Delta V_k / r_k'^2$, where $\cos \alpha_k''$ is intermediate between the maximum and the minimum of $\cos \alpha$ in ΔV_k and furthermore is a value which $\cos \alpha$ takes at some point p_k'' of ΔV_k. Hence the total x-component of the attraction of V for m is expressible in the form

$$X = \sum_{k=1}^{n} \kappa m \rho_k \cos \alpha_k'' \Delta V_k / r_k'^2 . \tag{2}$$

This would end our labors if we knew how to compute conveniently the coördinates of p_k, p_k', p_k''. But as we do not, it is necessary to take the limit of the sum on the right as δ approaches zero. According to the theorem of § 1 the value of this limit is also

$$X = \lim_{\delta=0} \sum_{k=1}^{n} \kappa m \rho_k \cos \alpha_k \, \Delta V_k / r_k^2 = \int_V \frac{\kappa m \rho \cos \alpha}{r^2} dV$$

found by changing p_k' and p_k'' both to p_k in the sum (2), α_k and r_k denoting the values of α and r at the point p_k.

The area of a surface generated by revolving a plane curve about the x-axis may be defined, in the well-known way, to be the limit of the sum of the lateral areas of certain inscribed cones. The sum of these lateral areas has the form

$$2\pi\sum_{k=1}^{n}\frac{f(x_{k-1})+f(x_k)}{2}\sqrt{1+[f'(x_k')]^2}\Delta x_k,$$

where the equation of the curve is $y = f(x)$ and x_k' s a suitably chosen value in the interval Δx_k of which x_{k-1} and x_k are the ends. According to the theorem of § 1, for regions of one dimension, the limit of this sum has the value

$$\lim_{\Delta x=0} 2\pi\sum_{k=1}^{n} f(x_k)\sqrt{1+[f'(x_k)]^2}\Delta x_k = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}dx.$$

The pressure of a liquid upon the plane wall of a containing vessel can be computed in a similar way. We admit (1) that on a horizontal area A at a depth x in a liquid there is a pressure equal to wxA, where w is the weight of a cubic unit of the liquid; (2) that upon any area B whatsoever there is a pressure greater than or equal to the pressure upon an equal horizontal

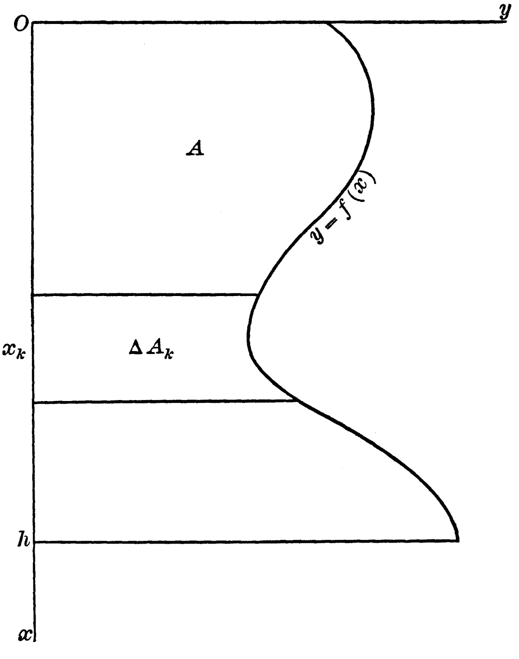

FIG. 1.

area at the depth of the highest point of B, and less than or equal to the pressure upon an equal horizontal area at the depth of the lowest point of B; and (3) that the pressure upon the sum of a number of areas is the sum of the pressures on the areas.

Let the area on which the pressure is to be computed be of the form shown in the accompanying figure, the y-axis being in the surface of the liquid. The function $f(x)$ is supposed to be continuous and the interval 0–h is divided into n parts Δx_k ($k = 1, 2, \cdots, n$). The area ΔA_k has the value

$$\Delta A_k = f(x_k)\Delta x_k,$$

where $f(x_k)$ is a suitably chosen ordinate between the smallest and the largest ordinates of the interval Δx_k. The pressure on ΔA_k has the value

$$\Delta P_k = wx_k'f(x_k)\Delta x_k,$$

where x_k' is a suitably chosen abscissa between the two which bound the interval Δx_k and define the highest and lowest points of ΔA_k. Hence the total pressure on the area A is

$$P = \sum_{k=1}^{n} wx_k'f(x_k)\Delta x_k.$$

As the intervals Δx_k are decreased in size the expression on the right is always equal to the pressure P, and, by the theorem of § 1 for a region of one dimension and with two points p_k and p_k' only instead of three, we have

$$P = \lim_{\Delta x = 0} \sum_{k=1}^{n} wx_kf(x_k)\Delta x_k = w\int_0^h xf(x)dx.$$

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